Topic 1 -
Vectors

Def: Let $n \geq 1$ be an integer.
[So, n can be $1, 2, 3, 4, 5, \ldots$]

An **n-dimensional real vector** is a list of n real numbers.
We use brackets $<$ and $>$ to denote vectors and commas to seperate the numbers. We use an arrow over a variable to denote a vector such as $\vec{v}$

Ex: Some 2-dimensional real vectors:

$\vec{v} = \langle 1, -2 \rangle$ ← order matters in the list of numbers so $\langle 1, -2 \rangle \neq \langle -2, 1 \rangle$

$\vec{w} = \langle \pi, \frac{1}{2} \rangle$

$\langle 0.126, 100 \rangle$

Ex: Some 4-dimensional real vectors:

$\langle 1, 1, 1, 1 \rangle$, $\langle 0, 0, 0, 0 \rangle$,

$\langle \pi, -\frac{2}{3}, e^2, \sqrt{2} \rangle$

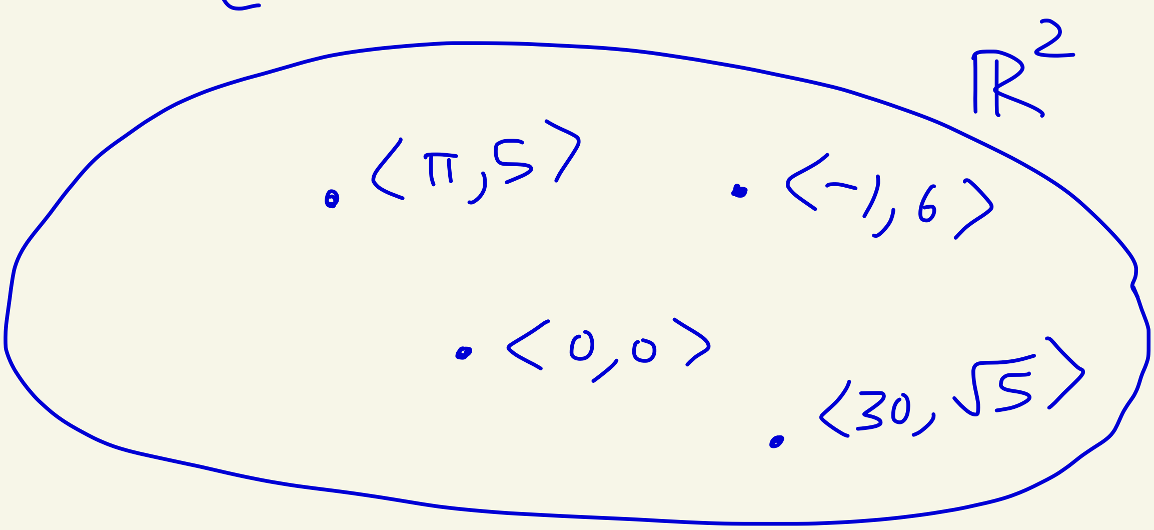
Def: Let $n \geq 1$ be an integer.
[So, n can be $1, 2, 3, 4, 5, \ldots$]
Define $\mathbb{R}^n$ to be the set of all n-dimensional real vectors.
That is,

$$\mathbb{R}^n = \{ \langle a_1, a_2, \ldots, a_n \rangle \mid a_1, a_2, \ldots, a_n \in \mathbb{R} \}$$

Ex:

$$\mathbb{R}^2 = \{ \langle a_1, a_2 \rangle \mid a_1, a_2 \in \mathbb{R} \}$$
$$= \{ \langle \pi, 5 \rangle, \langle -1, 6 \rangle, \langle 30, \sqrt{5} \rangle, \ldots \}$$

infinitely many more



Ex:

$$\mathbb{R}^3 = \{ \langle a_1, a_2, a_3 \rangle \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \{ \langle 0,0,5 \rangle, \langle 0,5,0 \rangle, \langle e, \pi, \sqrt{2222} \rangle, \langle 1, 2, \sqrt{2} \rangle, \ldots \}$$

infinitely many more

Ex:

$$\mathbb{R}^7 = \{ \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7 \rangle \mid a_1, a_2, \ldots, a_7 \in \mathbb{R} \}$$

$$= \{ \langle 0, 1, -1, \pi, 5, 7, 10 \rangle, \langle 1, 2, \pi, \sqrt{15}, \tfrac{1}{2}, e^2, 100 \rangle, \ldots \}$$

infinitely many more

Def: Let $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ be a vector in $\mathbb{R}^n$. (4)

Define the length (or norm or magnitude) of $\vec{v}$ to be

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

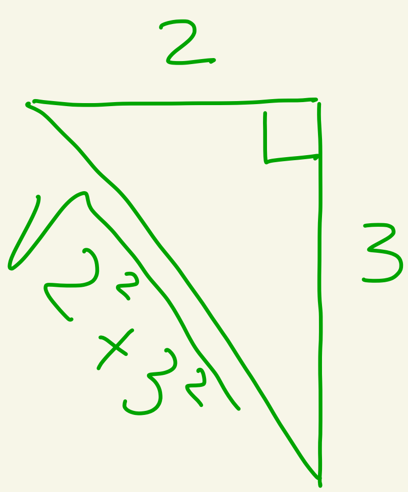
Some people write

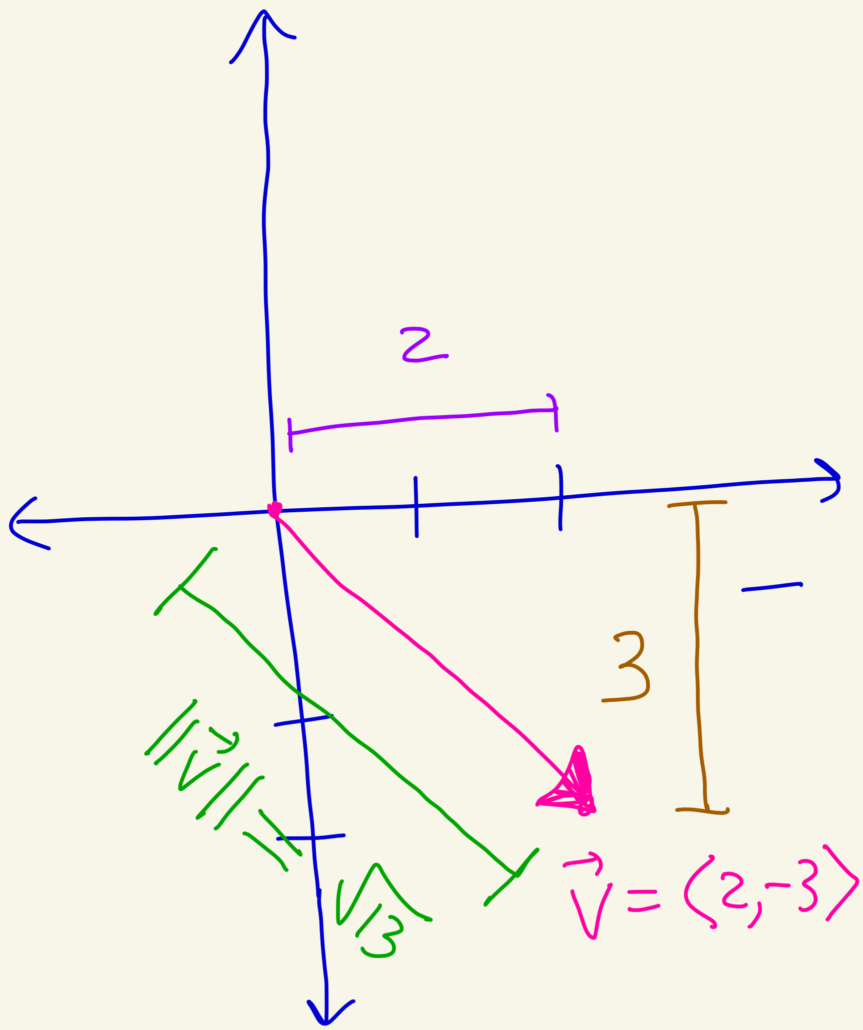
$$|\vec{v}|$$

instead of $\|\vec{v}\|$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, -3 \rangle$

Then, $\|\vec{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$





Ex: In $\mathbb{R}^7$,

let $\vec{w} = \langle 0, 1, -1, \frac{1}{2}, 0.1, 0, 2 \rangle$

Then,

$$\|\vec{w}\| =$$

$$= \sqrt{0^2 + 1^2 + (-1)^2 + \left(\tfrac{1}{2}\right)^2 + (0.1)^2 + 0^2 + 2^2}$$

$$= \sqrt{1 + 1 + \tfrac{1}{4} + 0.01 + 4}$$

$$= \sqrt{6 + 0.25 + 0.01}$$

$$= \sqrt{6.26}$$

Operations on vectors

Let $\vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$ and let α be a scalar in $\mathbb{R}$.

another word for number

Suppose

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$$

and

$$\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$$

some Greek letters

α alpha
β beta
γ gamma
δ delta

We define vector addition as follows:

$$\boxed{\vec{v} + \vec{w} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle}$$

We define scalar multiplication as follows:

$$\boxed{\alpha \vec{v} = \langle \alpha a_1, \alpha a_2, \ldots, \alpha a_n \rangle}$$

Subtraction is defined as

$$\boxed{\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle}$$

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 1, 3 \rangle$
and $\vec{w} = \langle -2, 1 \rangle$

Then,

$$\vec{v} + \vec{w} = \langle 1+(-2), 3+1 \rangle = \langle -1, 4 \rangle$$

$$2\vec{v} = 2\langle 1, 3 \rangle = \langle 2 \cdot 1, 2 \cdot 3 \rangle = \langle 2, 6 \rangle$$

$$\vec{v} - \vec{w} = \langle 1-(-2), 3-1 \rangle = \langle 3, 2 \rangle$$

$$-\vec{v} = (-1) \cdot \vec{v} = \langle -1, -3 \rangle$$

Ex: Let

$$\vec{v} = \langle 0, \tfrac{1}{2}, e, -1, 2, 6 \rangle$$

$$\vec{w} = \langle 1, 1, 1, 2, 2, 2 \rangle$$

be in $\mathbb{R}^6$.

Then,

$$\vec{v} + \vec{w} = \langle 0+1, \tfrac{1}{2}+1, e+1, -1+2, 2+2, 6+2 \rangle$$

$$= \langle 1, \tfrac{3}{2}, e+1, 1, 4, 8 \rangle$$

And

$$(-3)\vec{v} = \langle 0, -\tfrac{3}{2}, -3e, 3, -6, -18 \rangle$$

Also,

$$-3\vec{v} + \vec{w} = \langle 1, -\tfrac{1}{2}, -3e+1, 5, -4, -16 \rangle$$

<u>Notation:</u> In $\mathbb{R}^n$, the zero vector is the vector with all 0's. It is notated by $\vec{0}$.

In $\mathbb{R}^2$, $\vec{0} = \langle 0, 0 \rangle$

In $\mathbb{R}^3$, $\vec{0} = \langle 0, 0, 0 \rangle$

In $\mathbb{R}^4$, $\vec{0} = \langle 0, 0, 0, 0 \rangle$

And so on.

Properties of vectors

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$
and let α, β be in $\mathbb{R}$.

Then:

(commutative property)

(1) $\vec{u}+\vec{w} = \vec{w}+\vec{u}$

(associative property)

(2) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(3) $\alpha(\beta\vec{u}) = (\alpha\beta)\vec{u}$

(distributive property)

(4) $(\alpha+\beta)\vec{u} = \alpha\vec{u}+\beta\vec{u}$

(5) $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$

(6) $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$

(7) $\vec{u}+(-\vec{u}) = \vec{0}$

$(-\vec{u})+\vec{u} = \vec{0}$

α - alpha
β - beta

Proof of 2 when $n=3$:

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^3$.

Then,

$$\vec{u} = \langle a, b, c \rangle$$
$$\vec{v} = \langle d, e, f \rangle$$
$$\vec{w} = \langle g, h, i \rangle$$

where $a, b, c, d, e, f, g, h, i \in \mathbb{R}$.

We have that

$$(\vec{u} + \vec{v}) + \vec{w}$$
$$= \left(\langle a, b, c \rangle + \langle d, e, f \rangle \right) + \langle g, h, i \rangle$$
$$= \langle a+d, b+e, c+f \rangle + \langle g, h, i \rangle$$
$$= \langle (a+d)+g, (b+e)+h, (c+f)+i \rangle$$

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$$= \langle (a+d)+g, (b+e)+h, (c+f)+i \rangle$$

$$= \langle a+(d+g), b+(e+h), c+(f+i) \rangle$$

$(\alpha+\beta)+\gamma$
$= \alpha+(\beta+\gamma)$
if $\alpha, \beta, \gamma \in \mathbb{R}$
associative property of $\mathbb{R}$

$$= \langle a,b,c \rangle + \langle d+g, e+h, f+i \rangle$$

$$= \langle a,b,c \rangle + \left(\langle d,e,f \rangle + \langle g,h,i \rangle \right)$$

$$= \vec{u} + (\vec{v} + \vec{w})$$

Thus, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.

$\blacksquare$

Proof of (5) when $n=2$:

Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^2$.

Let α be in $\mathbb{R}$.

Then,

$$\vec{u} = \langle x, y \rangle$$

and $\vec{v} = \langle a, b \rangle$

where x, y, a, b are real numbers.

Then,

$$\begin{aligned}
\alpha(\vec{u} + \vec{v}) &= \alpha\left(\langle x, y \rangle + \langle a, b \rangle\right) \\
&= \alpha \langle x + a, y + b \rangle \\
&= \langle \alpha x + \alpha a, \alpha y + \alpha b \rangle.
\end{aligned}$$

same

We also have that

$$\begin{aligned}
\alpha \vec{u} + \alpha \vec{v} &= \alpha \langle x, y \rangle + \alpha \langle a, b \rangle \\
&= \langle \alpha x, \alpha y \rangle + \langle \alpha a, \alpha b \rangle \\
&= \langle \alpha x + \alpha a, \alpha y + \alpha b \rangle.
\end{aligned}$$

We see that $\alpha(\vec{u} + \vec{v}) = \alpha \vec{u} + \alpha \vec{v}$.

$\blacksquare$

Def: Let $\vec{v}$ and $\vec{w}$ be in $\mathbb{R}^n$ where $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{w} = \langle b_1, b_2, \ldots, b_n \rangle$.

The dot product of $\vec{v}$ and $\vec{w}$ is defined to be

$$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Note that $\vec{v} \cdot \vec{w}$ is a number

Ex: In $\mathbb{R}^2$, let $\vec{v} = \langle 2, -3 \rangle$ and $\vec{w} = \langle 4, 5 \rangle$.

Then,

$$\vec{v} \cdot \vec{w} = \langle 2, -3 \rangle \cdot \langle 4, 5 \rangle$$

$$= (2)(4) + (-3)(5)$$

$$= 8 - 15 = -7$$

Ex: In $\mathbb{R}^5$, let

$$\vec{v} = \langle 1, 2, 3, 4, 5 \rangle$$

and $\vec{w} = \langle -2, \frac{1}{2}, 1, \pi, \sqrt{2} \rangle$.

Then,

$$\vec{v} \cdot \vec{w}$$

$$= \langle 1, 2, 3, 4, 5 \rangle \cdot \langle -2, \frac{1}{2}, 1, \pi, \sqrt{2} \rangle$$

$$= (1)(-2) + (2)(\tfrac{1}{2}) + (3)(1)$$
$$+ (4)(\pi) + (5)(\sqrt{2})$$

$$= -2 + 1 + 3 + 4\pi + 5\sqrt{2}$$

$$= 2 + 4\pi + 5\sqrt{2}$$

Properties of the dot product

Let $\vec{u}, \vec{v}, \vec{w}$ be in $\mathbb{R}^n$ and let α be in $\mathbb{R}$.

Then:

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\alpha(\vec{u} \cdot \vec{v}) = (\alpha \vec{u}) \cdot \vec{v} = \vec{u} \cdot (\alpha \vec{v})$

Proof of property 2 when $n=2$:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$.

Then, $\vec{u} = \langle a, b \rangle$, $\vec{v} = \langle c, d \rangle$, and $\vec{w} = \langle e, f \rangle$ where a, b, c, d, e, f are real numbers.

We have that

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle a, b \rangle \cdot \left(\langle c, d \rangle + \langle e, f \rangle \right)$$

$$= \langle a, b \rangle \cdot \langle c+e, d+f \rangle$$

$$= a(c+e) + b(d+f)$$

$$= ac + ae + bd + bf$$

$$= \underbrace{ac + bd} + \underbrace{ae + bf}$$

$$= \underbrace{\langle a, b \rangle \cdot \langle c, d \rangle} + \underbrace{\langle a, b \rangle \cdot \langle e, f \rangle}$$

$$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

Therefore, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ $\blacksquare$

Ex: Do a set theory notation problem with vectors from HW 1. Do either problem 8 or 9.